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WEYL MANIFOLDS

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In 1918 H. Weyl [6] introduced a generalization of Riemannian geometry in his attempt to formulate a unified field theory. Weyl's theory failed for physical reasons, but it remains a beautiful piece of mathematics, and it provides an instructive example of non-Riemannian connections. In § 1 of this paper we summarize the classical definitions and theorems concerning Weyl structures; in § 2 we show that a Weyl structure is equivalent to a connection on a certain line bundle, prove the classical results using modern machinery and notation (following Kobayashi & Nomizu [4] and Nelson [5]), and derive a characterization of Weyl structures in terms of their induced linear connections.

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1. Summary of classical results

The physical motivation for Weyl's ideas is as follows. In the general theory of relativity, Einstein used Riemannian geometry as a model for physical space. However, the universe is not really a Riemannian manifold, for there is no absolute measure of length; that is, instead of being given a scalar product on the tangent space at each point, we are given a scalar product determined only up to a positive factor at each point. This fact produces no essential change in the geometry provided that a determination of length at one point uniquely induces a determination of length on the whole manifold, i.e., if it makes sense to compare the size of two tangent vectors at two distinct points. Weyl conjectured that this is not the case; rather, that an analogy should be drawn with the theory of linear connections, in which it generally makes sense to say that two vectors at two distinct points have the same direction only if there is specified a curve between the two points along which "parallel translation" can take place. Hence in the Weyl theory a determination of length at one point induces only a first-order approximation to a determination of length at surrounding points. We proceed to make these ideas precise.

M will always denote an n-dimensional smooth manifold, T_pM the tangent

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space of M at p, and $\Lambda^1(M)$ the space of one-forms on M.

Definitions.

1) Two Riemann metrics g and g' on M are said to be *equivalent* if and only if $g' = e^\lambda g$ where λ is a smooth function on M. (The use of the exponential function is a handy way of ensuring positivity, and it has other advantages which will become apparent later.)

2) A *conformal structure* on M is an equivalence class G of Riemann metrics on M. A manifold with a conformal structure is called a *conformal manifold*. Note that on a conformal manifold one can speak of the angle between two vectors at a point, or of the ratio of their lengths, although their absolute lengths are not defined. Also, the notion of a symmetric or skew-symmetric transformation of vector fields makes sense; for example, T is symmetric if and only if for any vector fields X and Y and for some (and hence all) $g \in G$, $g(TX, Y) = g(X, TY)$.

3) A *Weyl structure* on M is a map $F: G \to \Lambda^1(M)$ satisfying $F(e^\lambda g) = F(g) - d\lambda$, where G is a conformal structure. A manifold with a Weyl structure is called a *Weyl manifold*. Note that a Riemann metric g and a one-form φ determine a Weyl structure, namely F: $G \to \Lambda^1(M)$ where G is the equivalence class of g and $F(e^\lambda g) = \varphi - d\lambda$.

4) Since the forms in the range of a Weyl structure F differ from each other by exact forms, they have a common exterior derivative, the negative of which is denoted by Ω and is called the *distance curvature*.

5) A Weyl structure enables us to translate a scalar product along a curve, as follows. Let C: $[0, 1] \to M$ be a curve with $C(0) = p$, $C(1) = q$; let $(\ ,\ )_p$ be a scalar product on T_pM arising from the conformal structure G, and let $g \in G$ extend $(\ ,\ )_p$, i.e., $g_p = (\ ,\ )_p$. Then the *translate of* $(\ ,\ )_p$ *along* C *at* q is $(\ ,\ )_q = \exp\left[\int_0^1 C^*(F(g))\right] g_q$. This is independent of g: if $\lambda(p) = 0$, so that $(e^\lambda g)_p = (\ ,\ )_p$, then

$$\int_0^1 C^*(F(e^\lambda g)) = \int_0^1 C^*(F(g) - d\lambda) = \int_0^1 C^*(F(g)) - \lambda(q) ;$$

hence

$$\begin{aligned}(\ ,\ )_q &= \exp\left[\int_0^1 C^*(F(g))\right] g_q = \exp\left[\int_0^1 C^*(F(g)) - \lambda(q)\right] e^{\lambda(q)} g_q \\ &= \exp\left[\int_0^1 C^*(F(e^\lambda g))\right] (e^\lambda g)_q .\end{aligned}$$

Thus we can compare lengths of vectors at p and q.

6) A linear connection on a Weyl manifold M is said to be *compatible*

(with the Weyl structure) if and only if the parallel translation of scalar products arising from G by the linear connection is the same as their translation defined in 5).

The principal facts about Weyl manifolds, as discussed by Weyl [7, pp. 121–129] and Eisenhart [3, pp. 81–83], are the following. We state these results in the classical tensor notation, using the Einstein summation convention.

Let g be an arbitrary element of G, and denote $F(g)$ by φ.

A. A linear connection on a Weyl manifold M is compatible if and only if $g_{ij,k} + g_{ij}\varphi_k = 0$.

B. On every Weyl manifold there exists a unique torsion-free compatible linear connection; its components Γ^i_{jk} with respect to the coordinates $\{u^i\}_1^n$ are

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}\left(\frac{\partial g_{mj}}{\partial u^k} + \frac{\partial g_{km}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^m}\right) + \frac{1}{2}(\delta^i_j\varphi_k + \delta^i_k\varphi_j - g^{im}g_{jk}\varphi_m),$$

where (g^{ij}) is the inverse matrix of (g_{ij}), and δ^i_j is the Kronecker delta tensor.

C. Conversely, a torsion-free linear connection on a manifold M for which there exist a Riemann metric g and a one-form φ satisfying $g_{ij,k} + g_{ij}\varphi_k = 0$ is the induced connection of the Weyl structure determined by g and φ.

D. With respect to the natural connection, $g_{ij,km} - g_{ij,mk} = -2g_{ij}\Omega_{km}$.

E. If R^i_{jkm} is the curvature tensor of the natural connection, then $R^i_{jkm} = {}^*R^i_{jkm} + \delta^i_j\Omega_{km}$ where ${}^*R_{ijkm} = g_{ih}{}^*R^h_{jkm}$ is skew-symmetric in i and j.

These statements can be proved by laborious computation with indices. We will prove them with global methods in § 2.

Historical note. In Weyl's unified theory, the metrics $g \in G$ are interpreted as the gravitational potentials, as in general relativity, and the corresponding forms $F(g)$ are interpreted as the electromagnetic potentials. For more details on the physical theory see Weyl [6], [7, pp. 121 ff.], Eddington [2, pp. 196–240], and Adler et al. [1, pp. 401–417].

2. Weyl structures in terms of fiber bundles

A conformal structure G on M naturally gives rise to a fiber bundle $\mathscr{S}$ whose underlying manifold, also denoted by $\mathscr{S}$, is the set of all scalar products at all points of M arising from G, i.e., $\{g_p : g \in G, p \in M\}$, with the obvious differentiable structure, and whose projection $\pi : \mathscr{S} \to M$ is the map $\pi(g_p) = p$. This is clearly a principal fiber bundle over M whose structure group is the positive real multiplicative group $\boldsymbol{R}^*$. It is in fact a trivial bundle, since any $g \in G$ induces a trivialization $\sigma(g) : \mathscr{S} \to M \times \boldsymbol{R}^*$ defined by $\sigma(g)(g'_p) = (p, s)$ where $sg_p = g'_p \in \pi^{-1}(p)$. G can be recovered from $\mathscr{S}$ as the set of all smooth sections of $\mathscr{S}$.

We now fix an element g_0 of G, denote the trivialization $\sigma(g_0)$ simply by σ, and also define a function $r : \mathscr{S} \to \boldsymbol{R}$ by $r(s(g_0)_p) = s$.

There is a vector bundle $\mathcal{S}'$ associated with $\mathcal{S}$, whose underlying manifold is $\{s(g_0)_p: p \in M, s \in \boldsymbol{R}\}$, i.e., the set of real multiples of members of $\mathcal{S}$ with the obvious differentiable structure, and whose projection $\pi': \mathcal{S}' \to M$ is $\pi'(s(g_0)_p) = p$. Again we have a trivialization σ' given by $\sigma'(s(g_0)_p) = (p, s)$, and a function $r': \mathcal{S}' \to \boldsymbol{R}$ given by $r'(s(g_0)_p) = s$.

Definition. A *Weyl structure* on a conformal manifold M is a connection on the metric bundle $\mathcal{S}$.

We now have two definitions of a Weyl structure, and proceed to show that they are equivalent.

Theorem 1. *A connection on the metric bundle $\mathcal{S}$ of a conformal manifold M naturally induces a map $F: G \to \Lambda^1(M)$ with $F(e^\lambda g) = F(g) - d\lambda$, and conversely. Parallel translation of points in $\mathcal{S}$ by the connection is the same as their translation by F.*

Proof. Since the Lie algebra of $\boldsymbol{R}^*$ is $\boldsymbol{R}$, a connection form on $\mathcal{S}$ is real-valued. We will denote the action of $a \in \boldsymbol{R}^*$ on $\mathcal{S}$ by m_a.

Lemma 1. *ω is a connection form on $\mathcal{S}$ if and only if $\omega = \pi^* g_0^* \omega + r^{-1}dr$.*

Proof. Let $\bar{a}$ be the fundamental vector field on $\mathcal{S}$ corresponding to $a \in \boldsymbol{R}^*$. Since the left-invariant vector fields on $\boldsymbol{R}^*$ are of the form $ax(\partial/\partial x)$ where x is the restriction of the canonical coordinate on $\boldsymbol{R}$ to $\boldsymbol{R}^*$, it follows that $\bar{a} = ar(\partial/\partial r)$. ($(\partial/\partial r)$ is a well-defined vector field on $\mathcal{S}$ since $\mathcal{S} \cong M \times \boldsymbol{R}^*$.) Hence if ω is a form on $\mathcal{S}$ satisfying $\omega = \pi^* g_0^* \omega + r^{-1}dr$, then $\omega(\bar{a}) = r^{-1}ar = a$ since $\pi_* \bar{a} = 0$. Also,

$$m_a^* \omega = m_a^* \pi^* g_0^* \omega + (ar)^{-1} d(ar) = \pi^* g_0^* \omega + r^{-1}dr = \omega = (ad\, a)^{-1} \omega ,$$

since $\pi \circ m_a = \pi$ and $\boldsymbol{R}^*$ is commutative. Thus ω is a connection form. Conversely, if ω is a connection form, $\omega - \pi^* g_0^* \omega$ is some function f times dr, and the fact that $\omega(\bar{a}) = a$ for any $a \in \boldsymbol{R}^*$ shows that $f = r^{-1}$.

We return to the theorem. Given a connection on $\mathcal{S}$ with connection form ω, define $F: G \to \Lambda^1(M)$ by setting $F(e^\lambda g_0) = -g_0^* \omega - d\lambda$. Using the lemma, it is readily verified that F is independent of the choice of g_0, i.e., that $F(g) = -g^* \omega$ for any $g \in G$. Conversely. given $F: G \to \Lambda^1(M)$, define a one-form ω on $\mathcal{S}$ by $\omega = -\pi^*(F(g_0)) + r^{-1}dr$. By the lemma, ω is a connection form since $g_0^* \pi^* = id.$, and again it is easy to see that $\omega = -\pi^*(F(g)) + r^{-1}dr$ for any $g \in G$, so that ω is independent of g_0.

Now we show that translation by the connection and by F are equivalent. Let C be a curve on M and let $\tilde{C}$ be its horizontal lift to $\mathcal{S}$ with initial point $r_0(g_0)_{C(0)}$, so that $\tilde{C}(t)$ is the parallel-translate of $r_0(g_0)_{C(0)}$ along C to $C(t)$. Abbreviating $r(\tilde{C}(t))$ as $r(t)$, we have $\tilde{C}(t) = r(t)(g_0)_{C(t)} = m_{r(t)} \circ g_0 \circ C(t)$. Since ω is a vertical form, its restriction to $\tilde{C}$ is zero, i.e., $r^{-1}dr = -\pi^* g_0^* \omega$ on $\tilde{C}$. Thus

$$\int_0^t \tilde{C}^*(-\pi^* g_0^* \omega) = \int_0^t \tilde{C}^*(r^{-1}dr) = \log(r(t)/r_0) .$$

But $g_0{}^*m_{r(t)}{}^*\pi^* = g_0{}^*\pi^* = id.$, so that

$$C^*(-\pi^*g_0{}^*\omega) = -C^*(g_0{}^*m_{r(t)}{}^*\pi^*)g_0{}^*\omega = -C^*g_0{}^*\omega = C^*(F(g_0)) ,$$

so $\exp\left[\int_0^t C^*(F(g_0))\right] = r(t)/r_0$, which is the way translation by F was defined. Conversely, if C is a curve on M and $\bar{C}$ is the curve on $\mathscr{S}$ defined by $\bar{C}(t) = m_{r(t)} \circ g_0 \circ C(t)$ where $r(t) = \exp\left[\int_0^t C^*(F(g_0))\right]$, we have

$$\int_0^t \bar{C}^*(r^{-1}dr) = \log (r(t)/r_0) = \int_0^t C^*(F(g_0)) = -\int_0^t C^*g_0{}^*\omega$$
$$= -\int_0^t \bar{C}^*(g_0{}^*m_{r(t)}{}^*\pi^*)g_0{}^*\omega = -\int_0^t \bar{C}^*(\pi^*g_0{}^*\omega).$$

Thus $\int_0^t \bar{C}^*\omega = 0$ for each t, so $\bar{C}^*\omega = 0$. Hence $\bar{C}$ is horizontal and so is the unique horizontal lift of C to $\mathscr{S}$ whose initial point is $r_0(g_0)_{C(0)}$. This completes the proof.

We now investigate the most important consequence of the existence of a Weyl structure, namely, the existence of a unique torsion-free compatible linear connection.

Lemma 2 (Cf. **A** in § 1). *A linear connection on a Weyl manifold M is compatible if and only if $\nabla g + F(g) \otimes g = 0$ for all $g \in G$.*

Proof. For convenience, denote $F(g)$ by φ. Let X be a vector field on M, C an integral curve of X with $C(0) = p$, and τ_t the parallel translation of the tensor algebra from $C(0)$ to $C(t)$ by means of the linear connection. Compatibility then means that

$$\tau_t(g_p) = \exp\left[\int_0^t C^*\varphi\right]g_{C(t)}, \quad \text{or} \quad \tau_t^{-1}(g_{C(t)}) = \exp\left[-\int_0^t C^*\varphi\right]g_p .$$

Thus

$$(\nabla_X g)_p = \lim t^{-1}(\tau_t^{-1}g_{C(t)} - g_p) = \lim t^{-1}\left(\exp\left[-\int_0^t C^*\varphi\right] - 1\right)g_p$$
$$= \left((d/dt) \exp\left[-\int_0^t C^*\varphi\right]\Big|_{t=0}\right)g_p = -\varphi(X)_p g_p ,$$

so $\nabla_X g = -\varphi(X)g$ for any vector field X, i.e., $\nabla g + \varphi \otimes g = 0$. On the other hand, let $C(t)$ be a curve, and $X(t)$ its field of tangent vectors. Then g is parallel along C with respect to the linear connection if and only if $\nabla_X g = 0$, and it is parallel with respect to the Weyl structure if and only if $\varphi(X) = 0$. Thus if $\nabla g + \varphi \otimes g = 0$, these conditions are satisfied simultaneously.

Theorem 2 (Cf. **B** in § 1). *On every Weyl manifold there is a unique torsion-free compatible linear connection.*

Proof. Assume such a connection exists; we will derive an explicit formula for it and thus prove the uniqueness. Since ∇ is a derivation,

$$X(g(Y, Z)) = \nabla_X(g(Y, Z)) = (\nabla_X g)(Y, Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for any vector fields X, Y, and Z and any $g \in G$. Thus by compatibility, if $\varphi = F(g)$, then $g(\nabla_X Y, Z) = X(g(Y, Z)) + \varphi(X)g(Y, Z) - g(\nabla_X Z, Y)$. Also, zero torsion means $\nabla_X Y = [X, Y] + \nabla_Y X$, so we have

$$g(\nabla_X Y, Z) = X(g(Y, Z)) + \varphi(X)g(Y, Z) - g([X, Z], Y) - g(\nabla_Z X, Y) .$$

By cyclic permutation of X, Y, and Z, we obtain two more equations:

$$g(\nabla_Y Z, X) = Y(g(Z, X)) + \varphi(Y)g(Z, X) - g([Y, X], Z) - g(\nabla_X Y, Z)$$

and

$$g(\nabla_Z X, Y) = Z(g(X, Y)) + \varphi(Z)g(X, Y) - g([Z, Y], X) - g(\nabla_Y Z, X) .$$

Adding the first two and subtracting the third, we obtain

$$\begin{aligned} g(\nabla_X Y, Z) = \tfrac{1}{2}\{&X(g(Y, Z)) + \varphi(X)g(Y, Z) - g([X, Z], Y) \\ &+ Y(g(X, Z)) + \varphi(Y)g(Z, X) - g([Y, X], Z) \\ &- Z(g(X, Y)) - \varphi(Z)g(X, Y) + g([Z, Y], X)\} . \end{aligned} \tag{1}$$

Since g is a definite form, this equation determines $\nabla_X Y$.

To complete the proof, the following things must be checked: the right hand side of equation (1) is linear in Z and independent of the choice of g; ∇ satisfies the axioms for a connection, i.e., that it is linear in X and a derivation in Y; and ∇ is compatible and torsion-free. These verifications are just a matter of simple (if tedious) computation.

Note that the local coordinate expression for the connection given in **B** of § 1 is obtained by taking $X = \partial/\partial u^j$, $Y = \partial/\partial u^k$, $Z = \partial/\partial u^i$.

Corollary (Cf. **C** in § 1). *A torsion-free connection on a manifold M on which there exist a metric g and a one-form φ satisfying $\nabla g + \varphi \otimes g = 0$ is the induced connection of the Weyl structure determined by g and φ.*

Proof. It is easy to see that if g and φ satisfy $\nabla g + \varphi \otimes g = 0$ then so do $e^{\lambda} g$ and $\varphi - d\lambda$ for all smooth functions λ. The corollary then follows from Lemma 2 and the uniqueness assertion of Theorem 2.

Let ω and $D\omega$ be the connection and curvature forms, respectively, given by a Weyl structure on the metric bundle $\mathscr{S}$. Since multiplication in $\boldsymbol{R}$ is commutative, the structure equation $d\omega(X, Y) = \frac{1}{2}[\omega(X), \omega(Y)] + D\omega(X, Y)$ reduces to $d\omega = D\omega$. If Ω is the distance curvature and $g \in G$, then $\Omega = -d(F(g)) = -d(-g^*\omega) = g^*d\omega = g^*D\omega$. Thus the distance curvature on M is the pulldown of the curvature form on $\mathscr{S}$ via any $g \in G$.

If T is a tensor field of type (r, s) (i.e., of contravariant degree r and covariant degree s) on a manifold M with a linear connection, $\nabla\nabla T$ is a tensor field of type $(r, s+2)$ which thus assigns to each pair Y, Z of vector fields a tensor field $\nabla\nabla T(Y, Z)$ of type (r, s). If $R(Y, Z) = \nabla_Y\nabla_Z - \nabla_Z\nabla_Y - \nabla_{[Y,Z]}$ is the curvature transformation determined by the vector fields Y and Z, and X is any vector field, then the relation $\nabla\nabla X(Y, Z) - \nabla\nabla X(Z, Y) = R(Y, Z)X$ holds provided the connection is torsion-free (Nelson [5, p. 71]). On a Weyl manifold an analogous relation holds for the distance curvature Ω.

Lemma 3 (Cf. **D** in § 1). *For any $g \in G$ and any vector fields Y, Z,*

$$\nabla\nabla g(Y, Z) - \nabla\nabla g(Z, Y) = -2\Omega(Y, Z)g .$$

Proof. If $\varphi = F(g)$, we have

$$\begin{aligned}\nabla\nabla g(Y, Z) &= \nabla(-\varphi \otimes g)(Y, Z) = -(\nabla\varphi \otimes g)(Y, Z) - (\varphi \otimes \nabla g)(Y, Z) \\ &= -(\nabla\varphi \otimes g)(Y, Z) + (\varphi \otimes \varphi \otimes g)(Y, Z) = -(\nabla_Z\varphi)(Y)g + \varphi(Z)\varphi(Y)g .\end{aligned}$$

Thus

$$\begin{aligned}\nabla\nabla g(Y, Z) - \nabla\nabla g(Z, Y) &= ((\nabla_Y\varphi)(Z) - (\nabla_Z\varphi)(Y))g \\ &= 2d\varphi(Y, Z)g = -2\Omega(Y, Z)g ,\end{aligned}$$

since $d = \operatorname{alt} \nabla$ on forms (Nelson [5, p. 64]).

From this we can conclude a formula for the symmetric and skew-symmetric parts of the curvature transformation of the natural linear connection on a Weyl manifold. Let X, Y, Z, W be any vector fields, $R(X, Y)$ the curvature transformation determined by X and Y, $R(X, Y)^t$, its transpose (adjoint) and $\Phi_{R(X,Y)}$ the derivation of the tensor algebra induced by $R(X, Y)$. Using formulas (9) and (33) of Nelson [5, pp. 43, 71], we find that

$$\begin{aligned}-2g(\Omega(X, Y)Z, W) &= -2\Omega(X, Y)g(Z, W) = \nabla\nabla g(X, Y)(Z, W) \\ &\quad - \nabla\nabla g(Y, X)(Z, W) = (\Phi_{R(X,Y)}g)(Z, W) = -g(R(X, Y)Z, W) \\ &\quad - g(Z, R(X, Y)W) = -g([R(X, Y) + R(X, Y)^t]Z, W) ,\end{aligned}$$

for all W, and hence $\frac{1}{2}(R(X, Y) + R(X, Y)^t)Z = \Omega(X, Y)Z$ for all Z. Thus the symmetric part of $R(X, Y)$ is just $\Omega(X, Y)\delta$, δ being the Kronecker delta tensor of type $(1, 1)$. If we set $^*R = R - \Omega \otimes \delta$, then $^*R(X, Y)$ is the skew-symmetric part of $R(X, Y)$. $^*R$ is called the *direction curvature*. We sum up:

Theorem 3 (Cf. **E** in § 1). *For any vector vields X and Y, $\Omega \otimes \delta(X, Y) = \Omega(X, Y)\delta$ is the symmetric part of $R(X, Y)$, and $^*R(X, Y)$ is the skew-symmetric part.*

To prepare for the last theorem, we make some algebraic and bundle-theoretic remarks.

The *n-dimensional conformal group* $C(n)$ is the group of all real $n \times n$ matrices A such that $AA^t = kI$, where I is the identity matrix and k is a positive number. $C(n)$ is thus the set of all nonzero multiples of orthogonal matrices. Since the negative of an orthogonal matrix is orthogonal, $C(n)$ is isomorphic to $O(n) \times \boldsymbol{R}^*$. There is then a corresponding decomposition of the Lie algebra, $\mathfrak{c}(n) \cong \mathfrak{o}(n) \times \boldsymbol{R}$. Moreover, if $C(n)$ and $O(n)$ are regarded as subgroups of $GL(n, \boldsymbol{R})$, and $\mathfrak{c}(n)$ and $\mathfrak{o}(n)$ correspondingly as subspaces of $\mathfrak{gl}(n, \boldsymbol{R}) =$ the space of all $n \times n$ matrices, then $\mathfrak{o}(n)$ is the subspace of all skew-symmetric matrices, as is well known, and $\mathfrak{c}(n)$ is $\mathfrak{o}(n) \oplus \boldsymbol{R}I$. To verify this, let $A = B + kI$, where B is skew-symmetric and $k \in \boldsymbol{R}$. Then $\exp(tA) = \exp(tkI)\exp(tB) = e^{kt}\exp(tB)$. But $\exp(tB) \in O(n)$, so $e^{kt}\exp(tB) \in C(n)$. Thus $\mathfrak{o}(n) \oplus \boldsymbol{R}I \subset \mathfrak{c}(n)$, so they are equal since their dimensions are the same.

Now let M be a conformal manifold. A frame $\xi = \{\xi_i\}_1^n$ at a point $p \in M$ is said to be *conformal* if the vectors ξ_i are pairwise orthogonal and have equal lengths (although their absolute lengths are not determined). The bundle $\mathscr{C}$ of conformal frames on M is clearly a principal fiber bundle over M with structure group $C(n)$.

There is a natural bundle homomorphism h from the bundle $\mathscr{C}$ of conformal frames onto the metric bundle $\mathscr{S}$, which sends the frame ξ at $p \in M$ to the scalar product $(\ ,\ )_p$ at p with respect to which ξ is orthonormal. Now, if a connection is given on $\mathscr{C}$, there is induced a connection on $\mathscr{S}$ such that the horizontal subspaces over $\mathscr{C}$ are mapped onto the horizontal subspaces over $\mathscr{S}$; and if ω', ω are the connection forms, and $D\omega'$, $D\omega$ the curvature forms on $\mathscr{C}$ and $\mathscr{S}$ respectively, then $h^*\omega = H \circ \omega'$ and $h^*D\omega = H \circ D\omega'$ where H is the projection $\mathfrak{c}(n) \cong \mathfrak{o}(n) \times \boldsymbol{R} \to \boldsymbol{R}$ (Kobayashi & Nomizu [4, pp. 79–81]).

Theorem 4. *A torsion-free linear connection on a conformal manifold M is the induced connection of a Weyl structure on M if and only if it is reducible to the bundle $\mathscr{C}$ of conformal frames. In this case the Weyl structure is the connection on $\mathscr{S}$ induced via the homomorphism h of the preceding paragraph from the reduction of the linear connection to $\mathscr{C}$.*

Proof. Suppose M is a Weyl manifold. Then by compatibility, conformal frames are preserved under parallel translation, and it follows from the reduction theorem (Kobayashi & Nomizu [4, pp. 83–85]) that the linear connection is reducible to a connection on $\mathscr{C}$. To prove the last assertion of the theorem we must show that the homomorphism h maps the horizontal subspaces of the connection on $\mathscr{C}$ onto the horizontal subspaces of the Weyl structure on $\mathscr{S}$. If C is a curve on M, then its horizontal lift to $\mathscr{C}$ is a family $\tilde{C}$ of conformal frames, where $\tilde{C}(t) = \{\xi_i(t)\}_1^n$ is a frame over $C(t)$ and $\xi_i(t)$ is a parallel family of vectors for each i. Set $g_t = h \circ C(t)$. Then $g_t(\xi_i(t), \xi_j(t)) = \delta_{ij}$, so if $C'(t)$ is the tangent to C at t, then

$$0 = \nabla_{C'(t)}(g_t(\xi_i(t), \xi_j(t)) = (\nabla_{C'(t)}g)_t(\xi_i(t), \xi_j(t)) \\ - g_t(\nabla_{C'(t)}\xi_i(t), \xi_j(t)) - g_t(\xi_i(t), \nabla_{C'(t)}\xi_j(t)) = (\nabla_{C'(t)}g)_t(\xi_i(t), \xi_j(t)) \ .$$

Since $(\nabla_{C'(t)}g)_t$ is determined by its effect on the basis $\{\xi_i(t)\}$, it follows that g_t is a parallel family of scalar products along C. Thus by Theorem 1 and compatibility, g_t is a horizontal curve on $\mathscr{S}$. Since horizontal curves map to horizontal curves, horizontal subspaces map to horizontal subspaces, and the last assertion is proved.

On the other hand, suppose we are given a torsion-free linear connection on M which is reducible to $\mathscr{C}$. Let C be a curve on M, and τ_t parallel translation of the tensor algebra from $p = C(0)$ to $C(t)$. Reducibility to $\mathscr{C}$ means that $\mathscr{S}$ is preserved under τ_t (and hence under τ_t^{-1}). In fact, if g_p is in the fiber of $\mathscr{S}$ over p and $\{\xi_i\}$ is a frame at p, which is orthonormal with respect to g_p, then $\delta_{ij} = g_p(\xi_i, \xi_j) = (\tau_t g_p)(\tau_t\xi_i, \tau_t\xi_j)$ since τ_t is an isomorphism of the tensor algebra. But $\{\tau_t\xi_i\}$ is a conformal frame at $C(t)$, and since $\tau_t g_p$ is determined by its effect on a frame, it follows that $\tau_t g_p \in \mathscr{S}$. In particular, if $g_0, \mathscr{S}'$, and r' are as defined at the beginning of this section, $\tau_t^{-1}((g_0)_{C(t)}) \in \mathscr{S}$, so $\tau_t^{-1}((g_0)_{C(t)}) - (g_0)_p \in \mathscr{S}'$ and thus $(\nabla_{C'(t)}g_0)_p = \lim t^{-1}[\tau_t^{-1}((g_0)_{C(t)}) - (g_0)_p] \in \mathscr{S}'$. We therefore define a one-form φ by $\varphi(X) = -r'(\nabla_X g_0)$. By definition of r', $\nabla_X g_0 = -\varphi(X)g_0$ for all X, and so $\nabla g_0 + \varphi \otimes g_0 = 0$. By the corollary to Theorem 2, the proof is complete.

Remark. The decomposition of $R(X, Y)$ in Theorem 4 into a skew-symmetric transformation and a multiple of the identity corresponds to the decomposition of the curvature form $D\omega'$ of the reduced connection on $\mathscr{C}$ into its $\mathfrak{o}(n)$ and $\boldsymbol{R}I$ components in $\mathfrak{c}(n) \subset \mathfrak{gl}(n, \boldsymbol{R})$.

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